Pulsed Magnet Crimping

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1 Magnetic Crimping

Magnetoforming is a metal fabrication technique that has been in use for several decades. A large capacitor bank is used to store energy that is used to bend, weld, or otherwise work a piece of metal. In the following example, we will look at a magnetoforming apparatus. The apparatus uses a capacitor and a spark gap to transfer a large amount of energy into a work coil. A thin tube of Aluminum is placed in the work coil. The work coil exerts some force on the tube, and the tube is crimped radially around a plug placed in its open end (see diagram). Below is an electrical schematic of the apparatus, and a description of what we are going to do to analyze this machine.

- First, we will describe the action of the crimper,
- Second, we will derive an expression for the current as a function of time in the coil, the resonant frequency, the peak current, and the skin depth of the Aluminum tube with the emitted EM radiation,
- Last, our task remains to derive an expression for the pressure as a funtion of time, the peak pressure, and a simple expression for the peak pressure based on the physical parameters of the apparatus.

1. Crimping effect

   If we consider the Al tube as a sum of infinitessimal looks stacked up inside a larger sum of current loops (the inductor), we can use Faraday's law of induction and Lenz's law to explain the crimping effect.

   If we create a large magnetic induction B with the solenoid over a relatively short time period with a large $\frac{\partial I}{\partial t}$, we create a large $\frac{\partial \vec{B}}{\partial t}$. Faraday shows that

   $$\nabla \times E = -\frac{\partial \vec{B}}{\partial t} \tag{1}$$

   So an $\vec{E}$ field is created whose direction is such that an emf is induced in the tube. Assuming the tube is roughly the same size as the solenoid (all of the field goes through the tube), the induced emf in the wall of the tube will have the folowing form:

$$\mathcal{E} = -\mathcal{M}\frac{\partial I}{\partial t} \tag{2}$$

where the I is the changing current in the inductor, and $\mathcal{M}$ the mutual inductance of the system. This emf ($\mathcal{E}$) gives rise to the induced current. Note, however, that from $\nabla \times \vec{E}$, effectively Lenz's Law, the induced current flows in the opposite direction of the original current. This induced current leads to a new $\vec{B}$ field, opposing the original $\vec{B}$ field.

These two opposing inductions exert a repulsive force on each other. The force exerted by the solenoid on the tube will be much greater than the force exerted by the tube on the solenoid. Thus, the tube feels a force radially inward, giving rise to the crimping effect.

2. Current, Peak current, Frequency, Skin Depth, etc.

   The differential equation for this circuit has the following form:

$$V_o - R_b i_1 - \frac{1}{C}\int_0^t i_{1+2}dt - L\frac{\partial i_2}{\partial t} - R_s i_2 = 0 \tag{3}$$

   Where V_o is the initial charging voltage of the capacitor, i_1 is the current in loop 1, and i_2 is the current through loop 2. We have used Kirchoff's law and Ohm's law implicitly. At the peak value, the current through R_b is only 0.5mA, and the current through R_s is similarly derived to be several orders of magnitude higher, so we will neglect the effect of R_b in the differential equation to maintain sanity. Now, we have

$$V_o - R_s - \frac{1}{C}\int_0^t idt - L\frac{\partial i}{\partial t} = 0 \tag{4}$$

   With the caviat

$$L\frac{\partial^2 i}{\partial t^2} + R_s\frac{\partial i}{\partial t} + \frac{i}{C} = 0 \tag{5}$$

   The general solution is found to be

$$i(t) = e^{\frac{-R_s t}{2L}}\frac{2V_o}{\omega_0 L}\sinh\frac{R_s t}{2L}\sqrt{1-\frac{4L}{R_s^2 C}} \tag{6}$$

   But, we have $\frac{4L}{R_s^2 C} > 1$, so our general solution reduces to

$$i(t) = e^{\frac{-R_s t}{2L}}\frac{V_o}{\omega_o L}\sin\omega_o t \tag{7}$$

   Note that this has the form we would expect- a sine function with an exponential decay. Also notice that ω_o is the resonant frequency of the system. The resonant frequency has the following form:

$$\omega_o = \sqrt{\frac{1}{LC} - \frac{R_s^2}{4L^2}} \tag{8}$$

With the familiar relation $f = \frac{\omega_o}{2\pi}$. Plugging in the numbers given, we find the frequency of oscillation to be ω_o =181811.8$\frac{rad}{s}$ =28.94 kHz.

For the i_{max} we must take the derivative of $i(t)$ and set it to zero- giving us the time when the current reaches a maximum. Thus, we have

$$\frac{\partial i(t)}{\partial t} = 0 = \frac{-R_s}{2L} e^{\frac{-R_s t}{2L}} \frac{V_0}{\omega_0 L} \sin \omega_0 t + e^{\frac{-R_s t}{2L}} \frac{V_0}{L} \cos \omega_0 t \tag{9}$$

This gives us a time of 8.1×10^{-6} seconds, which corresponds to a maximum current of 1.6×10^4 Amps, or 16kA.

From the damping factor of a transmitted EM wave in a dispersive medium, i.e. $e^{-\beta z}$ with z the direction of propagation, we find that the distance the wave travels before falling to $\frac{1}{e}$ of its original value is given by $\Delta z = \beta^{-1}$, which is generally referred to as the skin depth, δ.

$$\delta = \frac{c}{\sqrt{2\pi\mu\omega_o\sigma}} \tag{10}$$

Above, c is the speed of light, μ is permiability of the material ($\mu\mu_0$, which in our case is simply the permiability of free space, and σ is the conductivity of the medium, or $\frac{1}{\rho}$. Plugging in the given numbers, we get δ =4.97x10^{-4} meters, or 0.5 mm.

3. Pressure as a function of time

   The magnetic pressure can be derived in a number of ways. I will outline two, as the answer was suprising to me (independent of the physical dimensions). First we will look at virtual work method. I will work in MKS units:

$$U_{magnetic} \equiv \int \frac{B^2}{2\mu_0} d\tau = \frac{B^2}{2\mu_0}(volume) = \frac{B^2}{2\mu_0}\pi r^2 l \tag{11}$$

   with l, and r being the physical dimentions of the surface we want to calculate the energy inside of. Let us call the $\hat{r}$ direction the radial direction. We know that force is defined as $\vec{F} \equiv -\nabla U$. Therefore, we can define a magnetic force in the following manner:

$$F_{magnetic} \equiv -\nabla U_{magnetic} = -\frac{\partial U_{magnetic}}{\partial r}\hat{r} = -\frac{B^2}{2\mu_0} 2\pi r l \tag{12}$$

   Notice that the force is in the $-\hat{r}$ direction. This means that the force is directed radially inward. So, the force will be pushing the tube in an inward, crushing manner. We can define a pressure

$$f_{magnetic} \equiv \frac{|F_{magnetic}|}{area} = \frac{|F_{magnetic}|}{2\pi rl} = \frac{B^2}{2\mu_0} \tag{13}$$

which is an interesting result.

Not believing it, I started from the stress tensor T_{ij}.

$$T_{ij} \equiv \epsilon(E_iE_j - \frac{1}{2}E^2\partial_{ij}) + \frac{1}{\mu_0}(B_iB_j - \frac{1}{2}B^2\partial_{ij}) \tag{14}$$

For our system $i = j$ and we are only interested in the B field, so T_{ij} reduces to:

$$T_{i=j} = -\frac{B^2}{2\mu_0} \tag{15}$$

Force is the time derivitive of momentum of the field, and is given as

$$\frac{\partial p_{i=j}}{\partial t} = \sum_j \int_v \frac{\partial}{\partial x_j} T_{i=j} d^3x \tag{16}$$

Applying the divergence theorem

$$\frac{\partial p_{i=j}}{\partial t} = \int_s \frac{\partial}{\partial x_j} T_{i=j} \cdot \hat{n} da \tag{17}$$

This gives us

$$F_{magnetic} = \int_s T_{i=j} \cdot \hat{n} da = -\frac{B^2}{2\mu_0}(area) \tag{18}$$

Which is the exact same result we got earlier. The magnetic pressure is independent of the actual size of the surface. The only place the physical dimention of the surface the pressure is exerted on is present in the $\vec{B}$ field.

Getting back to our problem, the magnetic field from an ideal solenoid is given as

$$\vec{B}_{solenoid} = \mu_0 n I(t)\hat{z} \tag{19}$$

where n is the number of turns per unit length, $n = \frac{N}{l}$. Here, the field is only dependent on the number of turns and the total length of the coil. Now we have the pressure as a function of time:

$$f_{magnetic} = \frac{\mu_0}{2} n^2 I^2(t_{max}) = \frac{\mu_0 n^2}{2} e^{\frac{-R_s t}{L}} \frac{V_o^2}{\omega_o^2 L^2} (\sin \omega_o t)^2 \tag{20}$$

And the peak pressure is calculated using the peak current as calculated above: $f_{peak} =$ 40 MPa = 5805 psi. This is plenty of pressure. It is certainly enough to crush Coke cans, if not more. Quarters placed in this solenoid would also probably shrink.

In order to get a good estimate for the actual pressure, we can make a few approximations. We know that the sin term of the pressure will reach maximum when the $t = t_{max}$. So, we let that equal 1. Also, we know that the exponential term will be very close to 1, since the t_{max} is so small. We are left with:

$$f_{magnetic} \approx \frac{\mu_0 n^2 V_o^2}{2\omega_o^2 L^2} \tag{21}$$

Now, since we are still making approximations we can say that $\omega_0 \approx (\sqrt{LC})^{-1}$. Plugging this in we have:

$$f_{magnetic} = \frac{\mu_0 n^2 V_o^2 C}{2L} = \frac{\mu_0 C}{2L}\left(\frac{N}{l}V_o\right)^2 = \frac{C}{2\pi l}\left(\frac{V_0}{R}\right)^2 \tag{22}$$

for a solenoid. Notice that the radius of the coil R is in the denominator. Any error in that measurment could yield very different pressures. Perhaps the better method of predicting the pressure is with the inductance measured directly, perhaps the next-to-last relation. Also, this is independent of the number of turns in the coil. To maximize pressure, one would have to minimize the radius first, then the length.